Final Exam - Introduction to Symplectic Geometry
M. Math II

03 May, 2025

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 105).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. (a) (5 points) Identify the Lie algebra, $\mathfrak{so}(3)^*$, of $SO(3,\mathbb{R}) = SO(3)$ with $\mathbb{R}^3$ as follows; define the map:

$$\mathbb{R}^3 \to T_eSO(3) : x = \left(x^1, x^2, x^3\right) \mapsto \hat{x} = \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix}$$

Show that $(\boldsymbol{x} \times \boldsymbol{y})^\wedge = [\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}]$, where $\times$ is the usual cross product on $\mathbb{R}^3$, and that $\boldsymbol{x} \cdot \boldsymbol{y} = -\frac{1}{2}$ trace $(\hat{\boldsymbol{x}}\hat{\boldsymbol{y}})$. (Thus the Lie algebra of $SO(3)$ may be viewed as $\mathbb{R}^3$ with cross product as Lie bracket.)

   (b) (10 points) Suppose $\hat{\boldsymbol{x}} \in T_eSO(3,\mathbb{R})$. Show that $\exp(t\hat{\boldsymbol{x}})$ is a rotation about the axis $\boldsymbol{x} \in \mathbb{R}^3$ through the angle $t\|\boldsymbol{x}\|$, where $\|\boldsymbol{x}\|$ is the Euclidean norm on $\mathbb{R}^3$.

   (c) (5 points) Consider the linear action $\Phi : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3 : (A, x) \mapsto Ax$. For $\xi \in \mathbb{R}^3$ let $\hat{\xi} \in T_eSO(3, R)$, where $^\wedge$ is given in (i), show that the infinitesimal generator is $\hat{\boldsymbol{\xi}}_{\mathbb{R}^3}(\boldsymbol{x}) = \boldsymbol{\xi} \times \boldsymbol{x}$.

(d) (5 points) Show that the adjoint action of $SO(3)$ on $\mathbb{R}^3$ is the "usual" action described in (iii).

(e) (5 points) The dual space $\mathfrak{so}(3)^*$ can be identified with $\mathbb{R}^3$ using the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$. Under this identification, For $R \in SO(3)$, show that the coadjoint action $Ad_R^* : \mathfrak{so}(3)^* \to \mathfrak{so}(3)^*$ corresponds to:

$$Ad_R^*(\mathbf{u}) = R\mathbf{u}.$$

Total for Question 1: 30

2. (10 points) Let (M, ω) be a symplectic manifold, where ω is a closed and non-degenerate 2-form. For a Hamiltonian function $H : M \to \mathbb{R}$, the Hamiltonian vector field X_H is defined by $i_{X_H}\omega = -dH$. Show that the flow ϕ_t of a Hamiltonian vector field X_H preserves the symplectic form, i.e., $\phi_t^*\omega = \omega$. (Hint: Use Cartan's magic formula)

Total for Question 2: 10

3. The Lie group $SE(2)$ represents rigid motions in the plane. Its Lie algebra $\mathfrak{se}(2)$ has a basis $\{P_x, P_y, J\}$ representing infinitesimal translations in the x and y directions, and infinitesimal rotation, respectively. The Lie bracket relations are given by $[J, P_x] = P_y$, $[J, P_y] = -P_x$, and $[P_x, P_y] = 0$. Let $\{p_x, p_y, j\}$ be the dual basis of $\mathfrak{se}(2)^*$. A covector $\xi \in \mathfrak{se}(2)^*$ can be written as $\xi = \mu p_x + \nu p_y + \lambda j$, where μ and ν correspond to linear momentum components, and λ to angular momentum.

(a) (10 points) For a general element $X = \omega J + v_x P_x + v_y P_y \in \mathfrak{se}(2)$ and a covector $\xi = \mu p_x + \nu p_y + \lambda j \in \mathfrak{se}(2)^*$, compute the infinitesimal coadjoint action $ad_X^*\xi$. Recall that $(ad_X^*\xi)(Y) = -\xi([X, Y])$ for any $Y \in \mathfrak{se}(2)$. Express the result as a linear combination of the basis elements $\{p_x, p_y, j\}$.

(b) (10 points) Consider a specific covector $\xi_0 = \lambda_0 j \in \mathfrak{se}(2)^*$, where λ_0 is a non-zero constant. Determine the coadjoint orbit $\mathcal{O}_{\xi_0} = \{Ad_g^*\xi_0 \mid g \in SE(2)\}$. Show that this orbit is a 2-dimensional submanifold of $\mathfrak{se}(2)^*$ and can be parameterized by the values of (μ, ν).

(c) (10 points) The tangent space to the coadjoint orbit $\mathcal{O}_{\xi_0}$ at a point $\eta \in \mathcal{O}_{\xi_0}$ is spanned by vectors of the form $ad_X^*\eta$ for $X \in \mathfrak{se}(2)$. Define the Kirillov-Kostant-Souriau symplectic form ω_{ξ_0} on this orbit by $\omega_{\xi_0}(ad_X^*\eta, ad_Y^*\eta) = \eta([X, Y])$. Compute this symplectic form explicitly in terms of the (μ, ν) coordinates on the orbit. Demonstrate that this form is non-degenerate. (A formal proof of closedness is not required here, focus on the structure of the form).

Total for Question 3: 30

4. The Lie group $SU(2)$ consists of 2×2 unitary matrices with determinant 1. Its Lie algebra $\mathfrak{su}(2)$ consists of 2×2 trace-zero skew-Hermitian matrices. A basis for $\mathfrak{su}(2)$ is given by $\{i\sigma_1/2, i\sigma_2/2, i\sigma_3/2\}$, where σ_j are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These satisfy the commutation relations $[i\sigma_j/2, i\sigma_k/2] = -\epsilon_{ijk}(i\sigma_l/2)$, where ϵ_{ijk} denotes the Levi-Civita symbol (0 if any two of i, j, k are the same, 1 if ijk is an even permutation of $(1, 2, 3)$, and -1 is ijk is an odd permutation of $(1, 2, 3)$).

We can identify $\mathfrak{su}(2)^*$ with $\mathfrak{su}(2)$ using the trace pairing. The coadjoint action is again $Ad^*_U(\xi) = U\xi U^\dagger$ for $U \in SU(2)$ and $\xi \in \mathfrak{su}(2)$. Any $\xi \in \mathfrak{su}(2)$ can be written as $\xi = i(a\sigma_1 + b\sigma_2 + c\sigma_3)/2$ with $a, b, c \in \mathbb{R}$.

(a) (5 points) Show that the eigenvalues of ξ are $\pm\frac{i}{2}\sqrt{a^2 + b^2 + c^2}$.

(b) (10 points) Describe the coadjoint orbits of $SU(2)$. What geometric shape do they correspond to? What happens when $a = b = c = 0$?

(c) (10 points) Compute the infinitesimal coadjoint action $ad^*_X\xi = [X, \xi]$ for $X = i\mathbf{x}\cdot\boldsymbol{\sigma}/2$ and $\xi = i\mathbf{j}\cdot\boldsymbol{\sigma}/2$, where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{j} = (a, b, c)$. Use the commutation relations of the Pauli matrices.

(d) (10 points) Compute the symplectic form $\omega_\xi([X, \xi], [Y, \xi]) = i\mathrm{tr}(\xi[X, Y])$ for ξ in a coadjoint orbit of $SU(2)$. Can you relate this form to the area form on the geometric shape you identified in part (b)?

Total for Question 4: 35